Applying TQFT to count regular coverings of Seifert 3-manifolds

Haimiao Chen *

Department of mathematics, Peking University, Beijing, China

Abstract

I give a formula for computing the number of regular Γ-coverings of closed orientable Seifert 3-manifolds, for a given finite group Γ. The number is computed using a 3d TQFT with finite gauge group, through a cut-and-glue process.

1   Introduction

The purpose of this article is to count regular coverings of closed orientable Seifert 3-manifolds, with a given finite covering group Γ; the problem is the same as counting homomorphisms from the fundamental groups to Γ.

The main results are the formulae in Theorem 3.3 and 4.1. They give an answer in terms of the conjugacy classes of Γ, the centralizers of elements of these classes and their characters.

A *Seifert 3-manifold* is a compact 3-manifold together with a decomposition into a disjoint union of circles (called *fibers*) such that, each fiber has a tubular neighborhood that is the mapping torus of an automorphism of a disk given by rotation by an angle of $2\pi b/a$ for a pair of coprime integers (a, b) with $a > 0$. A fiber with $a = 1$ (resp. $a > 1$) is called *ordinary* (resp. *exceptional*). The set of fibers forms a 2-dimensional orbifold called the *base-surface*. There are two types of connected closed orientable Seifert 3-manifolds according to whether the base-surface is orientable or not.

---

*Email: *chenhm@math.pku.edu.cn*

Seifert 3-manifolds form an important class of 3-manifolds. Most "small" 3-manifolds are Seifert manifolds, and they account for all compact oriented manifolds in 6 of 8 Thurston geometries of the Geometrization Conjecture. As interesting examples of Seifert 3-manifolds, there are Brieskorn complete intersections (see [11]) which include homology spheres as a subclass, the complements of torus knots in S^3 (see [10] Page 28), and so on.

The problem of enumerating finite-fold coverings of manifolds has been studied extensively in the past decades, especially in the last twenty years. For example, [7] gave formulae for regular coverings of surfaces, both orientable and non-orientable, both with and without boundary. There was much work on the realizability of branched coverings of surfaces, see, for instance, [12], [16]; the question is then whether the number of coverings of a certain kind is zero or not. [8] counted the numbers of homomorphisms from the fundamental groups of circle bundles over surfaces to permutation groups, so it actually enables us to count ordinary unbranched coverings of circle bundles over surfaces.

Generally speaking, the enumeration of isomorphism classes of (not necessarily connected) coverings of a space M is reduced to that of homomorphisms from $\pi_1(M)$ to a finite group. Namely, n-sheeted ordinary coverings of M correspond bijectively to homomorphisms $\pi_1(M) \to S_n$ (S_n is the permutation group on n letters); for a finite group Γ, regular Γ-coverings of M correspond bijectively to homomorphisms $\pi_1(M) \to \Gamma$. For more details see [6].

While there has been much work on enumerations in 2-dimensional topology, there is, besides [8], little on 3-dimensional topology, due to the complexity of 3-manifold groups.

Our results make some contributions to 3-dimensional enumeration. The approach uses TQFT with finite gauge group. Just as V.Turaev ([14], [15]) applied 2d TQFT to count representations of surface groups, we apply 3d TQFT to count representations of the fundamental groups of Seifert 3-manifolds. Compared with [8], we go further in two directions: the manifolds are allowed to have exceptional fibers, so they can be more complicated than just circle bundles; the target group may be any finite group, not just a permutation group.

The article is organized as follows. In Section 2, some basic notions and facts of TQFT are recalled. Section 3 and 4 are devoted to deriving formulae for enumeration. Some concrete computations are done in Section 5. The last section contains some remarks.

Notations.

Σ_g: the orientable closed surface of genus g.

$\Sigma_{g;p,q}$: the orientable surface of genus g, with boundary $(p+q)$ circles, p of which are oriented negatively, and the other q positively.

P: the pair of pants

ST: the solid torus $S^1 \times D$

$\#A$: the cardinality of a finite set A.

$C(x)$: the centralizer of an element x in a group.

2 TQFT with finite gauge group

In this section, we recall some notions and facts on TQFT. References are [2], [3], [4], [5]. Assume all manifolds are compact, smooth and oriented, and all maps are orientation-preserving diffeomorphisms.

2.1 Axioms of TQFT

For an integer l, as proposed by Freed (see [5]), an $(l+1)$-*dimensional topological quantum field theory* $((l+1)$d TQFT for short) is an assignment Z, assigning to each l-dimensional closed manifold Y a finite-dimensional Hermitian inner product space $Z(Y)$, with $Z(\emptyset) = \mathbb{C}$ (equipped with the standard inner product), and to each $(l+1)$-dimensional manifold X an element of the vector space $Z(\partial X)$, such that the following holds:

(a) (Functoriality) Every map $f : Y \to Y'$ induces an isometry

$$f_* : Z(Y) \to Z(Y');$$

and for $f : X \to X', f' : X' \to X''$,

$$f'_* \circ f_* = (f' \circ f)_*.$$

For every map $F : X \to X'$, one has

$$(\partial F)_*(Z(X)) = Z(X').$$

(b) (Orientation) There is a natural isometry

$$Z(-Y) \cong \overline{Z(Y)},$$

3

$(-Y$ has the same underlying manifold as Y but the opposite orientation) through which
$$Z(-X) = \overline{Z(X)}.$$

(c) (Multiplicativity) There is a natural isometry
$$Z(Y_1 \sqcup Y_2) \cong Z(Y_1) \otimes Z(Y_2),$$

through which
$$Z(X_1 \sqcup X_2) = Z(X_1) \otimes Z(X_2).$$

(d) (Gluing) If $Y^l \hookrightarrow X^{l+1}$ is a submanifold and X^{cut} is the manifold obtained by cutting X along Y, so that $\partial X^{cut} = \partial X \sqcup Y \sqcup -Y$, then
$$Z(X) = Tr_Y(Z(X^{cut})),$$

where
$$Tr_Y : Z(\partial X) \otimes Z(Y) \otimes \overline{Z(Y)} \to Z(\partial X)$$
is the contraction using the inner product on $Z(Y)$.

It follows from (b),(c) that when W is a cobordism from M_1 to M_2, that is, $\partial W = -M_1 \sqcup M_2$, then
$$Z(W) \in \overline{Z(M_1)} \otimes Z(M_2) \cong \hom(Z(M_1), Z(M_2)),$$

hence $Z(W)$ can be identified with a linear map $Z(M_1) \to Z(M_2)$. The axioms (a),(d) then tell us that Z is also a functor from the category of $(l+1)$-dimensional cobordisms to that of inner product spaces.

Also, (b) says that $Z(-W) : \overline{Z(M_2)} \to \overline{Z(M_1)}$ is the dual map of $Z(W)$.

Remark 2.1. $Z(X), Z(Y)$ *are called the* path-integrals *of* X, Y*, respectively.*

Remark 2.2. *The gluing axiom (d) also means that, when* $\partial X_1 = -Y, \partial X_2 = Y$*, and* X *is obtained by gluing* X_1, X_2 *along* Y *via a map* $f : Y \to Y$*, i.e.,* $X = (X_1 \sqcup X_2)/(y \in \partial X_1) \sim (f(y) \in \partial X_2)$*, then*
$$Z(X) = Tr_Y(Z(X_2) \otimes f_* Z(X_1)).$$

2.2 TQFT with finite gauge group

Given a finite group Γ and an integer l, there is a general method to construct an $(l+1)$d TQFT as follows; see [4], [5].

For a manifold M, let $\mathfrak{C}_M$ denote the groupoid of principal Γ-bundles over M with morphisms being the bundle morphisms covering the identity on M. Let $\overline{\mathfrak{C}}_M$ be the set of equivalence classes of these bundles.

For $P \in \mathfrak{C}_M$, setting

$$\mu(P) = \frac{1}{\#\mathrm{Aut}(P)}$$

defines a measure on $\mathfrak{C}_M$, and it descends to $\overline{\mathfrak{C}}_M$, by $\mu([P]) = \frac{1}{\#\mathrm{Aut}(P)}$ for any $P \in [P] \in \overline{\mathfrak{C}}_M$.

For $Q \in \mathfrak{C}_{\partial M}$, let $\mathfrak{C}_M(Q)$ be the groupoid of bundles $P \in \mathfrak{C}_M$ such that $\partial P = Q$, with morphisms being the bundle morphisms whose restriction to Q is the identity.

Let $\overline{\mathfrak{C}}_M(Q)$ be the set of equivalence classes. As above, one can define a measure on $\overline{\mathfrak{C}}_X(Q)$ by

$$\mu_Q([P]) = \frac{1}{\#\mathrm{Aut}_Q(P)},$$

where $\mathrm{Aut}_Q(P)$ is the group of automorphisms of P fixing Q.

For each closed l-dimensional manifold Y, set

$$Z(Y) = \mathrm{Map}(\overline{\mathfrak{C}}_Y, \mathbb{C}),$$

and equip it with the inner product

$$(f, g) = \sum_{[Q] \in \overline{\mathfrak{C}}_Y} \mu([Q]) f([Q]) \overline{g([Q])}.$$

For each $(l+1)$-dimensional manifold X, set

$$Z(X)([Q]) = \sum_{[P] \in \overline{\mathfrak{C}}_X(Q)} \mu_Q([P]) \in \mathbb{C}$$

for all $[Q] \in \overline{\mathfrak{C}}_{\partial X}$, where Q is any representative of $[Q]$. Thus $Z(X) \in Z(\partial X)$.

A more concrete description was given in Section 5 of [5]. Here we state it as a theorem.

Theorem 2.3. *Suppose X^{l+1}, Y^l are connected.*

$$Z(Y) = Map(\hom(\pi_1(Y), \Gamma)/\Gamma, \mathbb{C}), \tag{1}$$

where Γ acts by conjugation; for $\gamma : \pi_1(Y) \to \Gamma$, $\mu([\gamma]) = \frac{1}{\#C(\gamma)}$, with $C(\gamma)$ being the centralizer of $im\gamma \subset \Gamma$.
   If X is closed,

$$Z(X) = \frac{1}{\#\Gamma} \cdot \# \hom(\pi_1(X), \Gamma). \tag{2}$$

And if $\partial X \neq \emptyset$, for $\beta \in \hom(\pi_1(\partial X), \Gamma)$,

$$Z(X)([\beta]) = \#(\iota^*)^{-1}(\beta), \tag{3}$$

where $\iota^* : \hom(\pi_1(X), \Gamma) \to \hom(\pi_1(\partial X), \Gamma)$ is the restriction map.

Remark 2.4. *Such a TQFT is known as (untwisted)* Dijkgraaf-Witten the-ory, *named by the authors of [3], who first proposed it.*

2.3   (2+1)-dimensional DW theory

When $l = 2$, $E := Z(\Sigma_1)$ becomes the vector space of maps

$$\theta : \{(x, h) \in \Gamma \times \Gamma | xh = hx\} \to \mathbb{C} \tag{4}$$

satisfying

$$\theta(uxu^{-1}, uhu^{-1}) = \theta(x, h), \quad \forall u \in \Gamma; \tag{5}$$

and the inner product is given by

$$(\theta_1, \theta_2) = \frac{1}{\Gamma} \cdot \sum_{x,h} \theta_1(x, h)\overline{\theta_2(x, h)}. \tag{6}$$

By Lemma 5.4 of [5], E has a canonical orthonomal basis $\{\chi_i | i \in \Lambda\}$, where $\Lambda = \{i = (c, \rho)\}$, with c a conjugacy class of Γ and ρ an irreducible character of $C(x)$, the centralizer of x for an arbitrary choice of $x \in c$. We say that χ_i is *supported in the conjugacy class c* and denote

$$c = \text{supp}(\chi_i). \tag{7}$$

The Explicit expression of χ_i is

$$\chi_{(c,\rho)}(x,h) = \rho(h), \forall x \in c, h \in C(x); \quad \chi_{(c,\rho)}(x,h) = 0, \forall x \notin c. \tag{8}$$

Define

$$\dim \chi_{(c,\rho)} = \#c \cdot \rho(e). \tag{9}$$

Let P be the pair of pants (see Figure 1). Since $\partial(P \times S^1) = -\Sigma_1 \sqcup -\Sigma_1 \sqcup \Sigma_1$, we have

$$Z(P \times S^1) \in \overline{E} \otimes \overline{E} \otimes E \cong \hom(E \otimes E, E),$$

so it gives a product on E,

$$m : E \otimes E \to E,$$
$$m(\theta \otimes \vartheta)(x,h) = \sum_{x_1 x_2 = x} \theta(x_1, h)\vartheta(x_2, h). \tag{10}$$

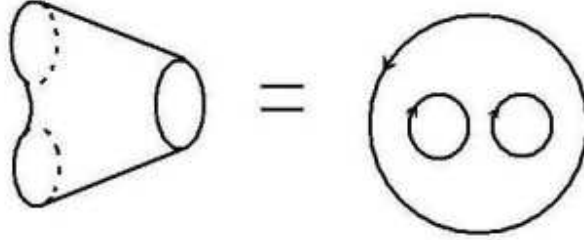The proof of formula (10) is similar to the proof of Proposition 5.17 of [5].

Figure 1: the pair of pants

The mapping class group of Σ_1, which is isomorphic to $SL(2, \mathbb{Z})$ (see [13], Page 22), acts on E by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \theta \right)_* (x,h) = \theta(x^a h^b, x^c h^d). \tag{11}$$

For the generators $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the expressions are (see [5], Proposition 5.8)

$$T_* \chi_i = \kappa_i \chi_i, \tag{12}$$
$$S_* \chi_i = \sum_{j \in \Lambda} s_i^j \chi_j, \tag{13}$$

7

where

$$\kappa_i = \chi_i(x, x)/\chi_i(x, e), \tag{14}$$

with $x \in \mathrm{supp}(\chi_i)$ chosen arrbitrarily, and

$$s_i^j = (S_* \chi_i, \chi_j) = \frac{1}{\#\Gamma} \cdot \sum_{x,h} \chi_i(h^{-1}, x)\overline{\chi_j(x, h)}. \tag{15}$$

In particular,

$$s_0^i = \frac{\dim \chi_i}{\#\Gamma}. \tag{16}$$

Suppose $m(\chi_i \otimes \chi_j) = \sum_k N_{ij}^k \chi_k$. By Proposition 3.1.12 of [2], if we define the matrices s, N_i, D_i respectively by

$$s_{ij} = s_i^j, \quad (N_i)_{jk} = N_{ij}^k, \quad (D_i)_{ab} = \delta_{ab}\frac{s_i^a}{s_0^a},$$

then $sN_i s^{-1} = D_i$. So setting

$$\tau_i = S_*^{-1} \chi_i \tag{17}$$

will diagonalize the product:

$$m(\tau_i \otimes \tau_j) = \frac{\delta_{ij}}{s_0^i}\tau_i. \tag{18}$$

Since S_*^{-1} is unitary, $\{\tau_i| \in \Lambda\}$ is still an orthonormal basis of E.

It is easy to see, by dualizing, that $Z(-P \times S^1)$ gives a coproduct

$$m' : E \to E \otimes E,$$
$$m'(\tau_i) = \frac{1}{s_0^i}\tau_i \otimes \tau_i. \tag{19}$$

For the solid torus which is denoted ST, let h indicate the longitude, and x the meridian.

8

Note that the homomorphism $\pi_1(T) \to \pi_1(ST)$ induced by the inclusion $T \hookrightarrow ST$ is a surjection whose kernel is generated by x, so we have

$$Z(ST)(x,h) = \delta_{x,e}, \quad (S_* Z(ST))(x,h) = \delta_{h,e},$$

and

$$Z(ST) = \sum_{i \in \Lambda} s_0^i \tau_i, \tag{20}$$

since $(Z(ST), \tau_i) = (S_* Z(ST), \chi_i) = s_0^i$.

Regarding $Z(-ST)$ as a morphism $E \to \mathbb{C}$,

$$Z(-ST)(\tau_i) = (\tau_i, Z(ST)) = \overline{s_0^i} = s_0^i. \tag{21}$$

3 Formula for Seifert 3-manifolds I: orientable base-surfaces

The strategy for determining $\# \hom(\pi_1(M), \Gamma)$ for a 3-manifold M is, relying on Theorem 2.3, to compute $Z(M)$.

3.1 Orientable Seifert 3-manifolds

According to [13], a closed orientable Seifert 3-manifold M can be obtained as follows. Take a circle bundle F which is orientable as a manifold, over a surface R with $\partial R = \sqcup_n S^1$, so that $\partial F = \sqcup_n \Sigma_1$, and glue n solid tori onto F along the boundary Σ_1's, via diffeomorphisms $f_j : \Sigma_1 \to \Sigma_1, j = 1, \cdots, n$. The "closure" of R, $\overline{R} = R \cup (\sqcup_n D)$, is the base-surface, and the images of the cores of ST, $S^1 \times 0 \subset ST$, are the exceptional fibers.

When f_j lies in the mapping class of Σ_1 represented by $\begin{pmatrix} a_j & b_j \\ u_j & v_j \end{pmatrix}$, denote M as $M(\overline{R}; (a_1, b_1), \cdots, (a_n, b_n))$.

In this section we assume $\overline{R}$ is orientable, $R = \Sigma_{g;n,0}$. Then F is diffeomorphic to $\Sigma_{g;n,0} \times S^1$. Denote M as $M_O(g; (a_1, b_1), \cdots, (a_n, b_n))$.

3.2 Computing $Z(\Sigma_{g;p,q} \times S^1)$

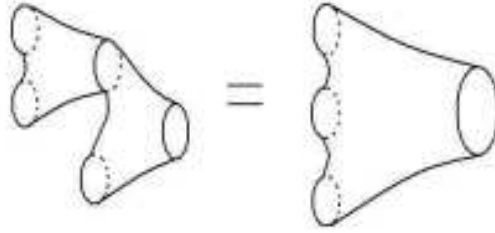Since $\Sigma_{1;1,1} = -P \cup_{S^1} P$, $Z(\Sigma_{1;1,1} \times S^1)$ is the composite

$$E \xrightarrow{m'} E \otimes E \xrightarrow{m} E, \quad \tau_i \mapsto (s_0^i)^{-2} \tau_i. \tag{22}$$

In general, since $\Sigma_{g;1,1}$ can be obtained by gluing g $\Sigma_{1;1,1}$'s successively, as shown in Figure 2(a), we have

$$Z(\Sigma_{g;1,1} \times S^1) = (m \circ m')^g : E \to E, \quad \tau_i \mapsto (s_0^i)^{-2g}\tau_i. \tag{23}$$

(a)

(b)

Figure 2: (a) $\Sigma_{g;1,1}$ obtained by gluing $\Sigma_{1;1,1}$'s; (b) $\Sigma_{0;p,1}$ obtained by gluing P's

For $p > 0$, $\Sigma_{0;p,1}$ can be obtained by gluing $(p-1)$ P's successively, (Figure 2(b) illustrates the case $p = 3$), hence $Z(\Sigma_{0;p,1} \times S^1) : E^{\otimes p} \to E$ is equal to

$$m \circ (1 \otimes m) \circ \cdots \circ (1 \otimes \cdots \otimes 1 \otimes m),$$
$$\tau_{i_1} \otimes \cdots \otimes \tau_{i_p} \mapsto (s_0^{i_1})^{1-p}\delta_{i_1,\cdots,i_p}\tau_{i_1}. \tag{24}$$

Dually, for $q > 0$, $Z(\Sigma_{0;1,q} \times S^1) : E \to E^{\otimes q}$ is equal to

$$(1 \otimes \cdots \otimes 1 \otimes m') \circ \cdots \otimes (1 \otimes m') \circ m', \quad \tau_i \mapsto (s_0^i)^{1-q}\tau_i^{\otimes q}. \tag{25}$$

For $p, q > 0$, $Z(\Sigma_{g;p,q} \times S^1) : E^{\otimes p} \to E^{\otimes q}$ is equal to the composite

$$E^{\otimes p} \xrightarrow{Z(\Sigma_{0;p,1} \times S^1)} E \xrightarrow{Z(\Sigma_{g;1,1} \times S^1)} E \xrightarrow{Z(\Sigma_{0;1,q} \times S^1)} E^{\otimes q},$$
$$Z(\Sigma_{g;p,q} \times S^1)(\tau_{i_1} \otimes \cdots \otimes \tau_{i_p}) = (\frac{1}{s_0^{i_1}})^{p+q+2g-2}\delta_{i_1,\cdots,i_p}\tau_{i_1}^{\otimes q}. \tag{26}$$

10

3.3 Considering exceptional fibers

Let $M'_O = M'_O(g; (a_1, b_1), \cdots, (a_n, b_n))$ be $M_O(g; (a_1, b_1), \cdots, (a_n, b_n))$ with an ST deleted. It is obtained by gluing n ST's onto $\Sigma_{g;n,1} \times S^1$, using $f_j : \Sigma_1 \to \Sigma_1, j = 1, \cdots, n$.

Since $(f_j)_*(Z(ST))(x, h) = \delta_{e, x^{a_j} h^{b_j}}$, we have

$$(f_j)_*(Z(ST)) = \frac{1}{\#\Gamma} \cdot \sum_{i \in \Lambda} \left(\sum_{\substack{xh = hx \\ x^{a_j} h^{b_j} = e}} \overline{\tau_i(x, h)} \right) \tau_i. \tag{27}$$

For any pair of coprime integers (a, b), define

$$\eta_i(a, b) = \sum_{\substack{xh = hx \\ x^a h^b = e}} \overline{\tau_i(x, h)}. \tag{28}$$

Lemma 3.1.

$$\eta_i(a, b) = \sum_{z \in \Gamma} \overline{\tau_i(z^{-b}, z^a)} = \sum_{z \in \Gamma} \chi_i(z^a, z^{-b}). \tag{29}$$

Proof. since $(a, b) = 1$, there exists a k such that $(ka - b, \#\Gamma) = 1$, (just let k be the product of all prime factors of $\#\Gamma$ that do not divide b). Take r such that $r(ka - b) \equiv 1 \pmod{\#\Gamma}$, then whenever $hx = xh, x^a h^b = e$, one has $h = h^{r(ka-b)} = h^{rka} x^{ra} = (xh^k)^{ra}$; let $y = xh^k$, we see $h = y^{ra}, x = yh^{-k} = y^{1-rka} = y^{-rb}$. Note that the map

$$\{(x, h) | xh = hx, x^a h^b = e\} \to \Gamma, \quad (x, h) \mapsto (xh^k)^r,$$

is bijective, with the inverse map given by $z \mapsto (z^{-b}, z^a)$. So the lemma is established. $\qquad \square$

Remark 3.2. *Suppose $(a, \#\Gamma) = d$. Taking c such that $ac \equiv d \pmod{\#\Gamma}$, we have*

$$\eta_i(a, b) = \sum_{z^{a/d} = y \in \Gamma} \chi_i(z^a, z^{-b}) = \sum_{y \in \Gamma} \chi_i(y^d, y^{-bc}). \tag{30}$$

In particular, when $(a, \#\Gamma) = 1$, $\eta_i(a, b) = \dim \chi_i \cdot \kappa_i^{-bc}$.

11

Going on, $Z(M_O')$ is the composite

$$\mathbb{C} \cong \mathbb{C}^{\otimes n} \xrightarrow{\overset{n}{\underset{j=1}{\bigotimes}}((f_j)_* Z(ST))} E^{\otimes n} \xrightarrow{Z(\Sigma_{g;n,1} \times S^1)} E.$$

$$Z(M_O')(1) = \sum_{i \in \Lambda} \left(\frac{1}{(\#\Gamma)^n (s_0^i)^{n+2g-1}} \cdot \prod_{j=1}^{n} \eta_i(a_j, b_j) \right) \tau_i. \qquad (31)$$

Finally, $Z(M_O(g; (a_1, b_1), \cdots, (a_n, b_n)))$ is the composite

$$\mathbb{C} \xrightarrow{Z(M_O')} E \xrightarrow{Z(-ST)} \mathbb{C},$$

$$Z(M_O(g; (a_1, b_1), \cdots, (a_n, b_n))) = \sum_{i \in \Lambda} \frac{(\#\Gamma)^{2g-2}}{(\dim \chi_i)^{n+2g-2}} \cdot \prod_{j=1}^{n} \eta_i(a_j, b_j). \qquad (32)$$

Thus we have the following

Theorem 3.3. *The number of regular Γ-coverings of the Seifert 3-manifold $M_O(g; (a_1, b_1), \cdots, (a_n, b_n))$ is*

$$\sum_{i \in \Lambda} \frac{(\#\Gamma)^{2g-1}}{(\dim \chi_i)^{n+2g-2}} \cdot \prod_{j=1}^{n} \eta_i(a_j, b_j). \qquad (33)$$

Remark 3.4. *Be careful that, by (2), $Z(M)$ is the number of homomorphisms $\pi_1(M) \to \Gamma$ divided by $\#\Gamma$.*

4 Formula for Seifert 3-manifolds II: non-orientable base-surfaces

When the base-surface $\overline{R}$ is non-orientable, $\overline{R}$ is the connected sum of some $\mathbb{R}P^2$'s. Denote $\Pi_g = \#_g \mathbb{R}P^2$. Let $\Pi_{g;n}$ be Π_g with n disks removed.

In the notation of Section 3.1, when $R = \Pi_{g;n}$, denote the Seifert manifold by $M_N(g; (a_1, b_1), \cdots, (a_n, b_n))$.

According to [8], over $\Pi_{g;1}$ there is up to isomorphism only one orientable circle bundle F_g, whose fundamental group has the presentation

$$\langle x, h, y_1, \cdots, y_g | x = \prod_{j=1}^{g} y_j^2, y_j h = h y_j, h^2 = e \rangle.$$

12

Noticing that $\Pi_{g,n} = \Pi_{g,1} \cup_{S^1} \Sigma_{0;n,1}$, it is then easy to see that the Seifert manifold $M_N(g; (a_1, b_1), \cdots, (a_n, b_n))$ can be obtained by gluing F_g with $M_O' = M_O'(0; (a_1, b_1), \cdots, (a_n, b_n))$ along Σ_1.

Since $\partial F_g = \Sigma_1$, $Z(F_g) \in E$. By (3), when $h^2 \neq e$, $Z(F_g)(x, h) = 0$; when $h^2 = e$, $Z(F_g)(x, h) = \#\{(y_1, \cdots, y_g) \in \Gamma^g | y_j h = h y_j, \prod_{j=1}^{g} y_j^2 = x\}$. Thus

$$(Z(F_g), \tau_i) = (S_* Z(F_g), \chi_i)$$

$$= \frac{1}{\#\Gamma} \cdot \sum_{\substack{xh=xh \\ x^2=e}} \#\{(y_1, \cdots, y_g) \in \Gamma^g | h \prod_{j=1}^{g} y_j^2 = e, y_j \in C(x)\} \overline{\chi_i(x, h)}.$$

$$(34)$$

But by [7] Theorem 4,

$$\#\{(y_1, \cdots, y_g) \in \Gamma^g | h \prod_{j=1}^{g} y_j^2 = e, y_j \in C(x)\}$$

$$= (\#C(x))^{g-1} \sum_{\chi} c_\chi^g \chi(e)^{1-g} \chi(h), \tag{35}$$

where χ ranges over the irreducible characters of $C(x)$, and c_χ is the Frobenius-Schur indicator (see [7]), which is defined for any character ρ of a group G,

$$c_\rho = \frac{1}{\#G} \cdot \sum_{y \in G} \rho(y^2). \tag{36}$$

Here for $j \in \Lambda$, we modify it to define by taking any $x \in \text{supp}(\chi_j)$,

$$\tilde{c}_j = \frac{\delta_{e,x^2}}{\#C(x)} \cdot \sum_{h \in C(x)} \chi_j(x, h^2). \tag{37}$$

Then

$$(Z(F_g), \tau_i) = \frac{1}{\#\Gamma} \sum_{x^2=e} \sum_{\substack{j \in \Lambda \\ x \in \text{supp}(\chi_j)}} \sum_{h \in C(x)} (\#C(x))^{g-1} c_{\chi_j}^g \chi_j(x, e)^{1-g} \chi_j(x, h) \overline{\chi_i(x, h)}$$

$$= \frac{\tilde{c}_i^g}{\#\Gamma} \cdot \sum_{x \in \text{supp}(\chi_i)} \#C(x) \left(\frac{\#C(x)}{\chi_i(x, e)}\right)^{g-1} = \left(\frac{\#\Gamma}{\dim \chi_i}\right)^{g-1} \cdot \tilde{c}_i^g, \tag{38}$$

13

where we have used the orthogonality relation $\sum\limits_{h \in C(x)} \chi_j(x,h)\overline{\chi_i(x,h)}$
$= \#C(x) \cdot \delta_{i,j}$, the identity $\#\mathrm{supp}(\chi_i) \cdot \#C(x) = \#\Gamma$ and (9).

Now $Z(M_N(g;(a_1,b_1),\cdots,(a_n,b_n)))$ is equal to the composite

$$\mathbb{C} \xrightarrow{Z(M'_O)} E \xrightarrow{(\cdot, Z(F_g))} \mathbb{C}.$$

By (31),(38), we obtain

$$Z(M_N(g;(a_1,b_1),\cdots,(a_n,b_n))) = \sum_{i \in \Lambda} \frac{(\#\Gamma)^{g-2}\tilde{c}_i^g}{(\dim \chi_i)^{n+g-2}} \prod_{j=1}^{n} \eta_i(a_j,b_j). \qquad (39)$$

Thus the enumeration for regular coverings of Seifert 3-manifolds with non-orientable base-surfaces is completed:

Theorem 4.1. *The number of regular Γ-coverings of the Seifert 3-manifold $M_N(g;(a_1,b_1),\cdots,(a_n,b_n))$ is*

$$\sum_{i \in \Lambda} \frac{(\#\Gamma)^{g-1}\tilde{c}_i^g}{(\dim \chi_i)^{n+g-2}} \prod_{j=1}^{n} \eta_i(a_j,b_j). \qquad (40)$$

5 Computations

Take $\Gamma = A_5$, the group of even permutations on 5 elements. It is the smallest nontrivial simple group.

In this section we compute the number of regular A_5-coverings of any orientable closed Seifert 3-manifold with orientable base-surface.

5.1 Facts on representation theory

The facts on A_5 which we recall below have been taken from [1] (Page 324).

There are 5 conjugacy classes:

$$[1], \quad [\alpha], \quad [\beta], \quad [\gamma], \quad [\gamma^2],$$

where

$$1 = (1), \quad \alpha = (12)(34), \quad \beta = (123), \quad \gamma = (12345),$$

and $[x]$ stands for the conjugacy class represented by x.

14

It is known that

$$(\#[1], \#[\alpha], \#[\beta], \#[\gamma], \#[\gamma^2]) = (1, 15, 20, 12, 12). \tag{41}$$

The corresponding centralizers are

$$C(1) = A_5, \quad C(\alpha) = \{(1), (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$
$$C(\beta) = \langle \beta \rangle \cong \mathbb{Z}/3\mathbb{Z}, \quad C(\gamma) = C(\gamma^2) = \langle \gamma \rangle \cong \mathbb{Z}/5\mathbb{Z}.$$

The irreducible characters of $C(1) = A_5$ are given by

$$\rho_1^1(1, \alpha, \beta, \gamma, \gamma^2) = (1, 1, 1, 1, 1), \tag{42}$$

$$\rho_2^1(1, \alpha, \beta, \gamma, \gamma^2) = (3, -1, 0, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}), \tag{43}$$

$$\rho_3^1(1, \alpha, \beta, \gamma, \gamma^2) = (3, -1, 0, \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}), \tag{44}$$

$$\rho_4^1(1, \alpha, \beta, \gamma, \gamma^2) = (4, 0, 1, -1, -1), \tag{45}$$
$$\rho_5^1(1, \alpha, \beta, \gamma, \gamma^2) = (5, 1, -1, 0, 0). \tag{46}$$

The other four centralizers are all abelian, and their irreducible characters are easy to find.

For $C(\alpha)$, set

$$\rho_1^\alpha \equiv 1, \quad \rho_2^\alpha(1, \alpha, (13)(24), (14)(23)) = (1, -1, 1, -1),$$
$$\rho_3^\alpha(1, \alpha, (13)(24), (14)(23)) = (1, 1, -1, -1), \quad \rho_4^\alpha = \rho_2^\alpha \rho_3^\alpha.$$

For $C(\beta)$, set

$$\rho_j^\beta(\beta) = e^{2(j-1)\pi i/3}, \quad 1 \leqslant j \leqslant 3.$$

For $C(\gamma)$, set

$$\rho_j^\gamma(\gamma) = e^{2(j-1)\pi i/5}, \quad 1 \leqslant j \leqslant 5.$$

For $C(\gamma^2)$, set

$$\rho_j^{\gamma^2} = \rho_j^\gamma, \quad 1 \leqslant j \leqslant 5.$$

These are all the irreducible characters.

At last, to be convenient, let

$$\Lambda = \{([x], \rho) | x = 1, \alpha, \beta, \gamma, \gamma^2, \rho = \rho_j^x\}. \tag{47}$$

And for $i = ([x], \rho) \in \Lambda$, define $\chi_i \in E$ by putting $\chi_i(y, g) = \rho(g)$ if y is conjugate to x and $g \in C(y)$, and putting $\chi_i(y, g) = 0$ otherwise.

5.2 Evaluating η

Now begin to evaluate $\eta_i(a, b)$ for any pair of coprime integers (a, b). Remark 3.2 will be referred to repeatedly.

For $1 \leqslant j \leqslant 5$ set

$$\omega_j(2) = \sum_{h \in [\alpha]} \rho_j^1(h) = 15\rho_j^1(\alpha), \tag{48}$$

$$\omega_j(3) = \sum_{h \in [\beta]} \rho_j^1(h) = 20\rho_j^1(\beta), \tag{49}$$

$$\omega_j(5) = \sum_{h \in [\gamma] \cup [\gamma^2]} \rho_j^1(h) = 12(\rho_j^1(\gamma) + \rho_j^1(\gamma^2)). \tag{50}$$

Then

$$\eta_{([1], \rho_j^1)}(a, b) = \rho_j^1(1) + \sum_{\{2,3,5\} \ni p \mid a} \omega_j(p). \tag{51}$$

To see this, note that $\forall h \in \Gamma$, $|h| \in \{1, 2, 3, 5\}$. Suppose $(a, 60) = d$, choose an integer c such that $ac \equiv d \pmod{60}$, as in Remark 3.1. If $2|d$ and $3, 5 \nmid d$, then

$$\eta_{([1], \rho_j^1)}(a, b) = \sum_{h^d = 1} \rho_j^1(h^{-bc}) = \rho_j^1(1) + \sum_{h \in [\alpha]} \rho_j^1(h^{-bc})$$

$$= \rho_j^1(1) + \sum_{h \in [\alpha]} \rho_j^1(h) = \rho_j^1(1) + \omega_j(2).$$

The other cases are dealt with similarly.

The other values of $\eta_i(a, b)$ can be worked out more easily:

$$\eta_{([\alpha], \rho_j^\alpha)}(a, b) = \delta_{1, a \pmod 2} \cdot 15 \cdot (-1)^{b(1-j)}, \quad 1 \leqslant j \leqslant 4, \tag{52}$$

$$\eta_{([\beta], \rho_j^\beta)}(a, b) = \delta_{1, a^2 \pmod 3} \cdot 20 \cdot (e^{2\pi i/3})^{ab(1-j)}, \quad 1 \leqslant j \leqslant 3, \tag{53}$$

$$\eta_{([\gamma], \rho_j^\gamma)}(a, b) = \delta_{1, a^4 \pmod 5} \cdot 12 \cdot (e^{2\pi i/5})^{a^3 b(1-j)}, \quad 1 \leqslant j \leqslant 5, \tag{54}$$

$$\eta_{([\gamma^2], \rho_j^{\gamma^2})}(a, b) = \delta_{1, a^4 \pmod 5} \cdot 12 \cdot (e^{2\pi i/5})^{2a^3 b(1-j)} \quad 1 \leqslant j \leqslant 5. \tag{55}$$

5.3 The result

The values of $\dim \chi_i$ for $i \in \Lambda$ are:

$$(\dim \chi_{([1],\rho_1^1)}, \dim \chi_{([1],\rho_2^1)}, \dim\chi_{([1],\rho_3^1)}, \dim \chi_{([1],\rho_4^1)}, \dim \chi_{([1],\rho_5^1)}) = (1,3,3,4,5); \tag{56}$$

$$\dim\chi_{([\alpha],\rho_j^\alpha)} = 15, \quad 1 \leqslant j \leqslant 4; \tag{57}$$

$$\dim\chi_{([\beta],\rho_j^\beta)} = 20, \quad 1 \leqslant j \leqslant 3; \tag{58}$$

$$\dim \chi_{([\gamma],\rho_j^\alpha)} = \dim \chi_{([\gamma^2],\rho_j^{\gamma^2})} = 12, \quad 1 \leqslant j \leqslant 5. \tag{59}$$

Below we simplify $\pmod p$ into (p).

$$\sum_{j=1}^{4} \frac{(\#\Gamma)^{2g-2}}{(\dim \chi_{([\alpha],\rho_j^\alpha)})^{n+2g-2}} \prod_{k=1}^{n} \eta_{([\alpha],\rho_j^\alpha)}(a_k, b_k)$$

$$= \sum_{j=1}^{4} 4^{2g-2} \left(\prod_{k=1}^{n} \delta_{1,a_k(2)}\right) (-1)^{(j-1)\sum\limits_{k=1}^{n} b_k}$$

$$= 4^{2g-1}\delta_{0,\sum\limits_{k=1}^{n} b_k(2)} \prod_{k=1}^{n} \delta_{1,a_k(2)}. \tag{60}$$

Similarly,

$$\sum_{j=1}^{3} \frac{(\#\Gamma)^{2g-2}}{(\dim \chi_{([\beta],\rho_j^\beta)})^{n+2g-2}} \prod_{k=1}^{n} \eta_{([\beta],\rho_j^\beta)}(a_k, b_k) = 3^{2g-1}\delta_{0,\sum\limits_{k=1}^{n} a_k b_k(3)} \prod_{k=1}^{n} \delta_{1,a_k^2(3)}, \tag{61}$$

$$\sum_{j=1}^{5} \frac{(\#\Gamma)^{2g-2}}{(\dim \chi_{([\gamma],\rho_j^\gamma)})^{n+2g-2}} \prod_{k=1}^{n} \eta_{([\gamma],\rho_j^\gamma)}(a_k, b_k) = 5^{2g-1}\delta_{0,\sum\limits_{k=1}^{n} a_k^3 b_k(5)} \prod_{k=1}^{n} \delta_{1,a_k^4(5)}, \tag{62}$$

$$\sum_{j=1}^{5} \frac{(\#\Gamma)^{2g-2}}{(\dim \chi_{([\gamma^2],\rho_j^{\gamma^2})})^{n+2g-2}} \prod_{k=1}^{n} \eta_{([\gamma^2],\rho_j^{\gamma^2})}(a_k, b_k) = 5^{2g-1}\delta_{0,\sum\limits_{k=1}^{n} a_k^3 b_k(5)} \prod_{k=1}^{n} \delta_{1,a_k^4(5)}. \tag{63}$$

Using (51) to compute the last term, and noting $\omega_2(p) = \omega_3(p)$, we have

$$\sum_{j=1}^{5} \frac{(\#\Gamma)^{2g-2}}{(\dim \chi_{([1],\rho_j^1)})^{n+2g-2}} \prod_{k=1}^{n} \eta_{([1],\rho_j^1)}(a_k, b_k) =$$

$$60^{2g-2} \prod_{k=1}^{n} \left(1 + \sum_{\{2,3,5\}\ni p | a_k} \omega_1(p)\right) + 2 \cdot 20^{2g-2} \prod_{k=1}^{n} \left(1 + \frac{1}{3} \sum_{\{2,3,5\}\ni p | a_k} \omega_2(p)\right) +$$

$$15^{2g-2} \prod_{k=1}^{n} \left(1 + \frac{1}{4} \sum_{\{2,3,5\}\ni p | a_k} \omega_4(p)\right) + 12^{2g-2} \prod_{k=1}^{n} \left(1 + \frac{1}{5} \sum_{\{2,3,5\}\ni p | a_k} \omega_5(p)\right).$$

$$(64)$$

Finally, $Z(M_O(g; (a_1, b_1), \cdots, (a_n, b_n)))$ is the sum of the expressions on the right-hand side of (60)-(64). Multiplying by 60 gives the number of regular A_5-coverings of the Seifert 3-manifold $M_O(g; (a_1, b_1), \cdots, (a_n, b_n))$.

6   Further remarks

1. As the last section illustrates, we can compute explicitly the number of regular Γ-coverings of any closed orientable Seifert 3-manifolds, as long as we know enough about Γ.

   We choose Γ to be A_5 in the example, because it is a non-solvable finite group, whence beyond the scope of [9].

2. In [8] the authors did not deal with exceptional fibers, because they would present additional difficulties in their approach. In our approach they are easy to deal with, thanks to the cut-and-glue property of TQFT.

3. In principle the same method can be used for computing regular Γ-coverings of general graph 3-manifolds, which can be obtained by gluing Seifert 3-manifolds along boundary tori. In that case the s-matrix, which can be complicated, will play a key role, because the gluing of the tori depends on the mapping class group of Σ_1.

References

[1] M.Artin. *Algebra*. China Machin Press, Beijing, 2004.

[2] B.Bakalov. *Lectures on tensor categories and modular functors*. University Lecture Series, 21. American Mathematical Society, Providence, Rhode Island, USA, 2001.

[3] R.Dijkgraaf, E.Witten. *Topological gauge theories and group cohomology*. Commutations in Mathematical Physics, 129, 393-429, 1990.

[4] K.Ferguson. *Link invariants associated to TQFT's with finite gauge groups*. Journal of knot theory and its ramifications, 2:1, 11-36, 1993.

[5] D.S.Freed, F.Quinn. *Chern-Simons theory with finite gauge group*. Communications in Mathematical Physics, 156, 435-472, 1993.

[6] A.Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[7] G.Jones. *Enumeration of homomorphisms and surface-coverings*. Quart.J.Math., 46:2, 485-507, 1995.

[8] V.Liskovets, A.Mednykh. *Enumeration of subgroups in the fundamental groups of orientable circle bundles over surfaces*. Communications in algebra, 28:4, 1717-1738, 2000.

[9] D.Matei, A.I.Suciu. *Counting homomorphisms onto finite solvable groups*. Jounal of algebra, 286, 161-186, 2005.

[10] T.S.Mrowka, P.S.Ozsváth. *Low dimensional Topology*. IAS/PARK City Mathematics Series Vol.15, 2009.

[11] W.D.Neumann, F.Raymond. *Seifert manifolds, plumbing, μ-invariant and orientation reversing maps*. Algebraic and Geometric Topology, Lecture Notes in Mathematics Vol.664. Springer-Verlag Berlin Heidelberg, Germany, 1978.

[12] E.Pervova. *On the existence of branched coverings between surfaces with prescribed branch data I*. Algebraic & Geometric Topology 6, 1957-1985, 2006.

[13] N.Saveliev. *Lectures on the topology of 3-manifolds.* De Gruyter text-book. Berlin; New York: de Gruyter, 1999.

[14] V.Turaev. *On certain enumeration problems in two-dimensional topology.* Math. Res. Lett. 16:3, 515-529, 2009.

[15] V.Turaev. *Dijkgraaf-Witten invariants of surfaces and projective representations of groups.* Journal of geometry and physics, 57, 2419-2430, 2007.

[16] H. Zheng. *Realizability of branched coverings of S^2.* Topology and its applications, 153, 2124-2134, 2006.